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Note

Lattice Paths in Regions with the Catalan Property

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If $\alpha = \{\alpha_0, \alpha_1, ..., \alpha_n\}$ and $\beta = \{\beta_0, \beta_1, ..., \beta_n\}$ are two non-decreasing sets of integers such that $\alpha_0 = 0 < \beta_0$, $\alpha_n < \beta_n = n$, and $\alpha_i < i < \beta_i$ for $1 \leqslant i \leqslant n-1$, let L denote the set of lattice points (p, q) such that $0 \leqslant p \leqslant n$ and $\alpha_p \leqslant q \leqslant \beta_p$. We determine all such regions L with the property that the number of lattice paths from $(0, 0)$ to (p, p) in L is the Catalan number $(p+2)^{-1}\binom{2p+2}{p+1}$ for $0 \leqslant p \leqslant n$.

Suppose $\alpha = \{\alpha_0, \alpha_1, ..., \alpha_n\}$ and $\beta = \{\beta_0, \beta_1, ..., \beta_n\}$ are two non-decreasing sequences of integers such that $\beta_0 > \alpha_0 = 0$, $\alpha_n < \beta_n = n$, and $\alpha_i < i < \beta_i$ for $1 \leqslant i \leqslant n-1$; we assume $n \geqslant 1$. Let $L = L(\alpha, \beta)$ denote the region determined by the lattice points (p, q) such that $0 \leqslant p \leqslant n$ and $\alpha_p \leqslant q \leqslant \beta_p$, and let $\mathscr{L}$ denote the collection of all such regions for $n = 1, 2, ...$. With each lattice point (p, q) in a region L we associate the number $N(p, q)$ of lattice paths from $(0, 0)$ to (p, q) in L, where each step in such a path consists of moving from a point (r, s) to one of the points $(r+1, s)$ or $(r, s+1)$.

Let S_n denote the region determined by the sequences $\alpha = \{0, 0, ..., 0\}$ and $\beta = \{1, 2, ..., n, n\}$ for $n = 1, 2, ...$. The region S_3 and the corresponding numbers $N(p, q)$ are given in Figure 1.

The Catalan numbers

$$\left\{\frac{1}{p+2}\binom{2p+2}{p+1} : p = 0, 1, ...\right\} = \{1, 2, 5, 14, ...\}$$

arise in many combinatorial problems (see, e.g., [1, 3, 4, 5]). A well-known result (see [2, p. 71]) implies that

$$N(p, p) = \frac{1}{p+2}\binom{2p+2}{p+1} \tag{1}$$

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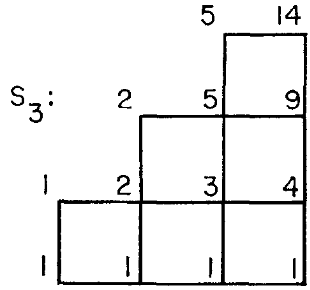

FIGURE 1.

for $0 \leqslant p \leqslant n$ for the region S_n, for $n = 1, 2, \ldots$. We shall say that a region L in $\mathscr{L}$ has the Catalan property if equation (1) holds for all diagonal points in L. The question arises whether there are regions that have the Catalan property other than the regions S_n and their reflections in the line $y = x$. Two such regions, which we denote by K_4 and K_8, and the corresponding numbers $N(p, q)$, are given in Figure 2. Our object here is to prove the following result.

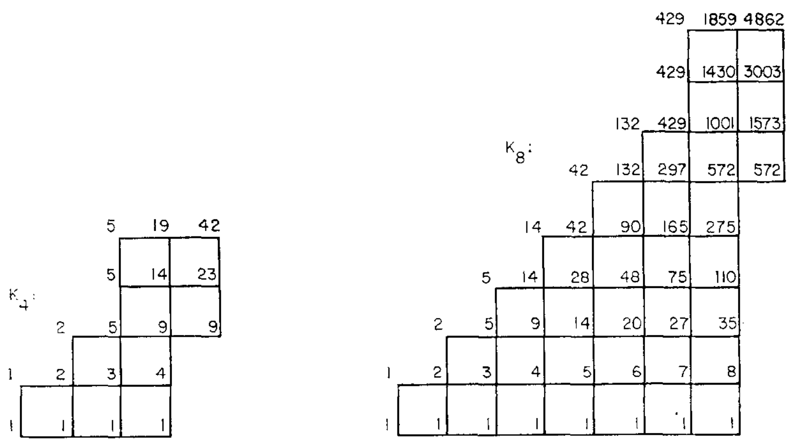

FIGURE 2.

THEOREM. *The only regions in $\mathscr{L}$ that have the Catalan property are $S_1, S_2, \ldots, K_4, K_8$, and the reflections of these regions in the line $y = x$.*

Proof. We first observe that the number $N(p, q)$ of lattice paths from $(0, 0)$ to (p, q) in the region S_n is given by the formula

$$N(p, q) = \frac{p - q + 2}{p + 2} \binom{p + q + 1}{q} \tag{2}$$

for $0 \leqslant p \leqslant n$ and $0 \leqslant q \leqslant \max\{p+1, n\}$. This follows by induction from the obvious recurrence relation

$$N(p, q) = N(p-1, q) + N(p, q-1)$$

where $N(r, s)$ is understood to be zero whenever the point (r, s) is not in S_n. (We remark that it is not difficult to see that the problem of determining $N(p, q)$ for the region S_n is equivalent to the well-known "Ballot Problem" described in [2, p. 70] and solved there by an argument based on the reflection principle.)

Let L_{n+1} denote any region in $\mathscr{L}$ with the Catalan property that contains the diagonal point $(n+1, n+1)$ but not the point $(n+2, n+2)$. If all points of the type $(n+1, q)$ and $(p, n+1)$ are removed from L_{n+1}, then the resulting region L_n clearly has the Catalan property also. We say that the region L_{n+1} is an extension of L_n. For example, the region S_{n+1} is an extension of S_n for $n = 1, 2, \ldots$, and the regions K_4 and K_8 are extensions of S_3 and S_7. We now suppose that the region L_{n+1} is an extension of S_n; we shall show this implies that L_{n+1} must be S_{n+1}, K_8, K_4, or the region obtained by reflecting S_2 in the line $y = x$.

If $\alpha_{n+1} = s$ for L_{n+1} then L_{n+1} consists of the points of S_n plus the points $(n+1, s), (n+1, s+1), \ldots, (n+1, n+1)$ as well as the point $(n, n+1)$ and, perhaps, the point $(n-1, n+1)$. (The regions K_4 and K_8 are examples of extensions containing the point $(n-1, n+1)$ for which s equals 2 and 5, respectively.) If we iterate the recurrence relation for $N(p, q)$ we find that

$$\begin{aligned} N(n+1, n+1) &= N(n, n+1) + N(n+1, n) \\ &= N(n, n) + N(n, n) + N(n+1, n-1) = \cdots \\ &= N(n, n) + N(n, n) + N(n, n-1) + \cdots \\ &\quad + N(n, s) + N(n+1, s-1). \end{aligned} \tag{3}$$

Let $M(p, q)$ denote the number of lattice paths from $(0, 0)$ to (p, q) in L_{n+1}; then $M(p, q) = N(p, q)$ for those points (p, q) that are in S_n. If we bear in mind the relation between L_{n+1} and S_n (the reader may find it helpful to make a sketch), we find that

$$\begin{aligned} M(n+1, n+1) &= M(n, n+1) + M(n+1, n) \\ &= M(n, n+1) + N(n, n) + M(n+1, n-1) = \cdots \quad (4) \\ &= M(n, n+1) + N(n, n) + N(n, n-1) + \cdots + N(n, s). \end{aligned}$$

If the point $(n-1, n+1)$ is not in L_{n+1} then $M(n, n+1) = M(n, n) = N(n, n)$; thus if

$$M(n+1, n+1) = N(n+1, n+1) = \frac{1}{n+3}\binom{2n+4}{n+2}, \tag{5}$$

then in this case (3) and (4) imply that $N(n+1, s-1) = 0$. That is, $s = 0$ and $L_{n+1} = S_{n+1}$, by definition. Furthermore, if $s = 0$, then (3), (4), and (5) imply that $M(n, n+1) = N(n, n)$. If this is the case then the point $(n-1, n+1)$ cannot be in L_{n+1} so we again conclude that $L_{n+1} = S_{n+1}$.

We may assume, therefore, that $1 \leqslant s \leqslant n$ and that $(n-1, n+1)$ is in L_{n+1}. Conequently,

$$\begin{aligned} M(n, n+1) &= M(n-1, n+1) + M(n, n) \\ &= N(n-1, n) + N(n, n) \\ &= N(n-1, n-1) + N(n, n). \end{aligned}$$

This relation and equations (3), (4) and (5) imply that

$$N(n-1, n-1) = N(n+1, s-1),$$

or that

$$\frac{1}{n+1}\binom{2n}{n} = \frac{n-s+4}{n+3}\binom{n+s+1}{s-1},$$

by equation (2). If we let $s = n - k$, where $0 \leqslant k \leqslant n-1$, then this equation can be rewritten as

$$(n+3)(n+2)(2n)_k = (2n+1-k)(n-k)(k+4)(n)_k, \tag{6}$$

where we adopt the convention that $(x)_0 = 1$ and $(x)_k = x(x-1) \cdots (x-k+1)$ for $k = 1, 2, \ldots$.

Since $(2n)_k \geqslant 2^k (n)_k$ and $(n+3)(n+2) > \frac{1}{2}(2n+1-k)(n-k)$ for $k \geqslant 0$, it follows that there are no solutions to (6) when $2^{k-1} \geqslant k+4$, that is, when $k \geqslant 4$. If we consider the remaining cases $k = 0$, 1, 2, and 3 individually we find that the only admissible solutions (n, k) to equation (6) are (7, 2), (3, 1), and (1, 0). The first two solutions correspond to the extensions K_8 and K_4, respectively; the last corresponds to the region obtained by reflecting S_2 in the line $y = x$.

It is not difficult to verify directly that no extension of K_4 or K_8 has the Catalan property. Let $\mathscr{C}$ denote the set consisting of the regions $S_1, S_2, \ldots, K_4, K_8$, and the reflections of these regions in the line $y = x$. The foregoing observations imply that if L_{n+1} is an extension of any region in $\mathscr{C}$ and L_{n+1} has the Catalan property, then L_{n+1} is also in $\mathscr{C}$. The required result now follows by induction on n, since every region L_{n+1} that has the Catalan property is an extension of some region L_n that has the Catalan property, when $n \geqslant 1$, and the region S_1 is clearly the only region with the Catalan property when $n = 1$.

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